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THE EFFECT OF MHD ON UNSTEADY FLOWS OF BURGER'S FLUID BETWEEN TWO SIDE WALLS PERPENDICULAR TO A PLATE THROUGH A POROUS MEDIUM

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ABSTRACT

This paper presents a research for the MHD flow of an incompressible generalized fractional Burgers' fluid between two side walls perpendicular to the plate in porous medium .The solutions of the velocity field are established by means of mixed Fourier sine transform and discrete Laplace transform in term of the generalized Mittag –Leffler function written as a direct sum of the Newtonian solution and the corresponding non-Newtonian solutions. Furthermore, the solution for generalized second grad fluid , generalized Maxwell fluid and generalized Oldroyd-B fluid are also obtained as limiting cases of our general solution .Finally , some characteristics of the motion as well as the influence of the material parameter on the behavior of the fluid are shown by graphical illustrations.

Keywords: *MHD, Burger's Fluid, Porous Medium etc.*

1. INTRODUCTION

The flows of non-Newtonian fluids are of very important in a number of engineering application ,such as the extrusion of polymer fluids, exotic lubricant ,animal bloods ,heavy oils and colloid and suspension solutions[8] . No model alone that can be described the behavior of all non –Newtonian fluids because of complex behavior . An important class of non-Newtonian fluids viscolastic fluids .The Burgers' fluid is a kind of them which cannot be described by a typical relation between shear stress and the rate of strain. Therefore, many models of constitutive equations have been applied for these fluids [4, 6,14 ,15,17].

Fractional calculus has been used much success in the description of viscoelasticity ,Recently ,many researchers have been proposed different problems related to anon-Newtonian fluids with fractional derivative . Fetecau et al.[1] , studied numerical fractional –calculus model of the Rayleigh Stokes problem for a Maxwell fluid. Ali et al. [5], discussed New exact solutions of Stokes' second problem for an MHD second grade fluid in a porous space . Hayet et al. [16] ,discussed the unsteady flow of a second grade fluid between two side walls perpendicular to a plate . Kang et al. [8] discussed thermal convective instability of viscoelastic fluids in a rotating porous layer heated from below. Khan [12], discussed Flow of a generalized second-grade fluid between two side walls perpendicular to a plate with a fractional derivative model. Jianhang Kang et al.[8] ,studied the unsteady flows of a Generalized Burgers' fluid between two side walls perpendicular to a plate with a fractional derivative model.

In this paper ,we study the effect of MHD on of an incompressible generalized fractional Burgers' fluid between two side walls perpendicular to the plate in porous medium. The velocity fields is determined by means of Laplace and mixed Fourier sine transform and are presented under integral and series forms in the Mittag –leffler function.

2. STATEMENT OF THE PROBLEM

Consider an incompressible Burgers' fluid occupying the space above an infinite flat plate and between two side walls perpendicular to this plate, as shown in Figure 1. The side walls are extended to infinity in the x- and y-directions and are located at $z = 0$ and $z = d$. Flow Induced by the Impulsive Motion of the Plate. Initially, the

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fluid is at rest and then the plate is suddenly brought to a steady velocity U_0 at the moment $t = 0^+$, corresponding initial-boundary conditions are given by

$u(y,z,0) = 0 \ , y > 0 \ , 0 \le z \le d$

$u(0,z,t) = U_0 \ , 0 < z < d \ \ , t > 0$

$u(y,0,t) = u(y,d,t) = 0 \ , y > 0 \ , t > 0$

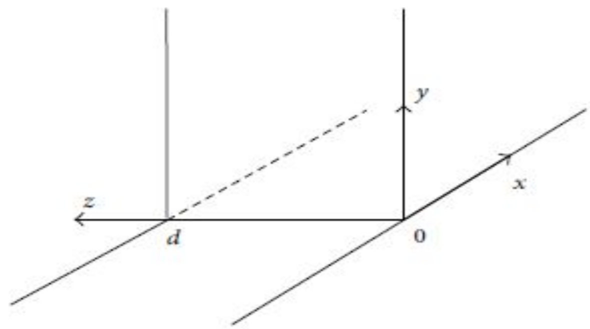

Figure 1: The schematic diagram of system considered here.

3. GOVERNING EQUATIONS

The continuity and momentum equations for an unsteady incompressible flow is governing by

$$\nabla . \ V = 0 \tag{1}$$

$$\rho \frac{dV}{dt} = -\nabla p + \nabla . S + J \times B + R \tag{2}$$

where ρ is the density of the fluid , V is the velocity, ∇ is the gradient operator , p is the pressure , S the extra stress tensor ,($J \times B$) is the component of Lorentz force (electronmagnetic) , B is the magnetic field , J is current density (or conduction current) and R is a measure of the flow resistance offered by the solid matrix.

The constitutive equations for an unsteady incompressible fractional Burgers' fluid given by

$$T = -pI + S \ , \left(1 + \lambda_1^{\alpha} \frac{\delta^{\alpha}}{\delta t^{\alpha}} + \lambda_2^{\alpha} \frac{\delta^{2\alpha}}{\delta t^{2\alpha}}\right) S = \mu \left(1 + \lambda_3^{\beta} \frac{\delta^{\beta}}{\delta t^{\beta}}\right) A_1 \tag{3}$$

Where T is the Cauchy stress tensor , μ is the dynamic viscosity , $A_1 = \nabla V + \nabla V^T$ is the first Rivlin -Ericksen tensor , λ_1 and $\lambda_3 (< \lambda_1)$ are relaxation and retardation times with the dimension of time, and λ_2 is the new material parameter of the Burgers' fluid , α and β the fractional calculus parameters such that $0 \le \alpha \le \beta \le 1$ and $\frac{\delta^{\alpha}}{\delta t^{\alpha}}$ the upper convected fractional derivative defined by

$$\frac{\delta^{\alpha} S}{\delta t^{\alpha}} = \frac{\partial^{\alpha} S}{\partial t^{\alpha}} + V . \nabla S - \nabla V . S - S . (\nabla V)^T \tag{4}$$

$$\frac{\delta^{\beta} A_1}{\delta t^{\beta}} = \frac{\partial^{\beta} A_1}{\partial t^{\beta}} + V . \nabla A_1 - \nabla V . A_1 - A_1 . (\nabla V)^T \tag{5}$$

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Where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional derivative of order α with respect to t ,which is defined as [10]

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_0^t (t-\tau)^{k-\alpha-1} f(\tau)\,d\tau \ ,(k-1 \le \alpha \le k) \tag{6}$$

Where $\Gamma(.)$ denotes the Gamma function and

$$\frac{\delta^{2\alpha} S}{\delta t^{2\alpha}} = \frac{\delta^\alpha}{\delta t^\alpha}\left(\frac{\delta^\alpha S}{\delta t^\alpha}\right) \tag{7}$$

We consider the MHD flow of an incompressible generalized Burgers' fluid due to an infinite flat plate and between two side walls perpendicular to this plate through porous medium .Assuming The velocity and extra stress tensor of stress tensor of fluids under consideration should have the forms:

$$V = [u(y,z,t), 0\ , 0] \quad , S = S(y,z,t) \tag{8}$$

in the Cartesian coordinate system, where $u(y,z,t)$ is the velocity component in the x-direction.

According to Eq,(8) , the continuity equation (1) is automatically satisfied .

For the components of the stress field S ,we get

$$S_{xy} = S_{yx} \ , S_{xz} = S_{zx} \ \text{ and } \ S_{zy} = S_{yz} \tag{9}$$

Substitute Eqs.(13) , (18) and (19) into Eq.(3), we get

$$S_{xy} + \lambda_1^\alpha\left(\frac{\partial^\alpha S_{xy}}{\partial t^\alpha} - u_y S_{yy} - u_z S_{yz}\right) + \lambda_2^\alpha\left(\frac{\partial^{2\alpha} S_{xy}}{\partial t^{2\alpha}} - u_y S_{yy} - u_z S_{yz}\right) = \mu\left(1 + \lambda_3^\beta \frac{\partial^\beta}{\partial t^\beta}\right) u_y \tag{10}$$

$$S_{xz} + \lambda_1^\alpha\left(\frac{\partial^\alpha S_{xz}}{\partial t^\alpha} - u_y S_{yz} - u_z S_{zz}\right) + \lambda_2^\alpha\left(\frac{\partial^{2\alpha} S_{xz}}{\partial t^{2\alpha}} - u_y S_{yz} - u_z S_{zz}\right) = \mu\left(1 + \lambda_3^\beta \frac{\partial^\beta}{\partial t^\beta}\right) u_z \tag{11}$$

$$\left(1 + \lambda_1^\alpha \frac{\partial^\alpha}{\partial t^\alpha} + \lambda_2^\alpha \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) S_{yy} = 0 \tag{12}$$

$$\left(1 + \lambda_1^\alpha \frac{\partial^\alpha}{\partial t^\alpha} + \lambda_2^\alpha \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) S_{yz} = 0 \tag{13}$$

$$\left(1 + \lambda_1^\alpha \frac{\partial^\alpha}{\partial t^\alpha} + \lambda_2^\alpha \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) S_{zz} = 0 \tag{14}$$

From Eqs.(12),(13) and (14) ,we obtain

$$S_{yy} = S_{yz} = S_{zz} = 0 \tag{15}$$

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Substitute Eq.(15) into Eqs.(10) and (11) ,we get

$$\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)S_{xy}=\mu\left(1+\lambda_3^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)u_y \quad (16)$$

$$\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)S_{xz}=\mu\left(1+\lambda_3^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)u_z \quad (17)$$

3.2 Momentum and Continuity Equations:

We will write the momentum equation which governing the magnetohydrodynamic in the porous medium

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial S_{xy}}{\partial y}+\frac{\partial S_{xz}}{\partial z}-\sigma B_0^2u-\frac{\mu\phi}{K}u \quad (18)$$

Elimination of S_{xy} and S_{xz} from Eqs.(16) and (17) ,yeilds

$$\rho\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\frac{\partial u}{\partial t}=-\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\frac{\partial p}{\partial x}+\mu\left(1+\lambda_3^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\times$$

$$\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\sigma B_0^2+\frac{\mu\phi}{K}\right)u \quad (19)$$

Divided by ρ ,we get

$$\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\frac{\partial u}{\partial t}=-\frac{1}{\rho}\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\frac{\partial p}{\partial x}+\upsilon\left(1+\lambda_3^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\times$$

$$\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(M+\upsilon\frac{\phi}{K}\right)u \quad (20)$$

Where $\upsilon=\frac{\mu}{\rho}$ is the kinematics' viscosity and $M=\frac{\sigma B_0^2}{\rho}$ is the magnetic number .

3.3 Solution of Unsteady Flows

Let us consider the flow Induced by the Impulsive Motion of the Plate. Initially, the fluid is at rest and then the plate is suddenly brought to a steady velocity U_0 at the moment $t = 0^+$. Such a motion is termed as the Rayleigh-Stokes' problem in the literature. In this case the governing equation , in the absence gradient in the flow direction ,is given by

$$\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\frac{\partial u}{\partial t}=\upsilon\left(1+\lambda_3^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$-\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\lambda_2^{\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(M+\upsilon\frac{\phi}{K}\right)u \quad (21)$$

The associated initial and boundary condition are follows :

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$$u(y,z,0) = 0 \ , y > 0 \ , 0 \le z \le d \tag{22}$$

$$u(0,z,t) = U_0 \ , 0 < z < d \ \ , t > 0 \tag{23}$$

$$u(y,0,t) = u(y,d,t) = 0 \ , y > 0 \ , t > 0 \tag{24}$$

Employing the non-dimensional quantities

$$u^* = \frac{u}{U_0} \ , \ t^* = \frac{t}{d/U_0} \ , (x^*, y^*, z^*) = \frac{(x, y, z)}{d} \ , \ \lambda_1^* = \lambda_1^\alpha \left(\frac{U_0}{d}\right)^\alpha \ , \ \lambda_2^* = \lambda_2^\alpha \left(\frac{U_0}{d}\right)^{2\alpha}$$

$$, \ \lambda_3^* = \lambda_3^\beta \left(\frac{U_0}{d}\right)^\beta \quad , Re = \frac{U_0 \, d}{\upsilon} \ , M^* = \frac{M \, d^2}{\upsilon \, Re} \ , \ \left(\upsilon \frac{\phi}{K}\right)^* = \frac{\phi}{K} \frac{d^2}{Re} \tag{25}$$

Where $\lambda_1^*, \lambda_2^*, \lambda_3^*$ and Re are the are the dimensionless relaxation time, material parameter, retardation time, Reynolds number and respectively.

Eqs.(21)-(25) in dimensionless form are :

$$\left(1 + \lambda_1 \frac{\partial^\alpha}{\partial t^\alpha} + \lambda_2 \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) \frac{\partial u}{\partial t} = \frac{1}{Re} \left(1 + \lambda_3 \frac{\partial^\beta}{\partial t^\beta}\right) \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
$$-\left(1 + \lambda_1 \frac{\partial^\alpha}{\partial t^\alpha} + \lambda_2 \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) \left(M + \upsilon \frac{\phi}{K}\right) u \tag{26}$$

$$u(y,z,0) = 0 \ , y > 0 \ , 0 \le z \le 1 \tag{27}$$

$$u(0,z,t) = 1 \ , 0 < z < 1 \ \ , t > 0 \tag{28}$$

$$u(y,0,t) = u(y,1,t) = 0 \ , y > 0 \ , t > 0 \tag{29}$$

It should be noted that in order to solve a problem for (26) additional conditions apart from (27)–(29) are supposed to be attached, that is

$$\frac{\partial u(y,z,0)}{\partial t} = \frac{\partial^2 u(y,z,0)}{\partial t^2} = 0 \ , y > 0 \ , 0 \le z \le 1 \tag{30}$$

$$u(y,z,t) \ , \ \frac{\partial u(y,z,t)}{\partial t} \to 0 \text{ as } y \to \infty \ , t > 0 \tag{31}$$

Where the dimensionless mark" *" has been omitted for simplicity.

The velocity field can be obtained by solving the governing equation (26) subject to conditions (26)-(30) .We will use the mixed Fourier sine transform [8] i.e

$$\bar{u}(\xi, \zeta_n, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^1 u(y,z,t) \sin y\,\xi \sin z\,\zeta_n \, dz dy \quad , n = 1,2,3 \ldots \tag{32}$$

And its inverse is

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$$u(y,z,t) = 2\sqrt{\frac{2}{\pi}}\int_0^\infty \sum_{n=1}^{\infty} \bar{u}(\xi,\zeta_n,t)\, \sin y\,\xi\, \sin z\,\zeta_n \; d\xi \quad , n = 1,2,3 \ldots .. \qquad (33)$$

In which $\zeta_n = n\pi$.

Now, applying the mixed Fourier sine transform to Eqs.(26) and taking into account boundary conditions (28),(29) and (31) ,we get

$$\left(1+\lambda_1\frac{\partial^\alpha}{\partial t^\alpha}+\lambda_2\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\frac{\partial\bar{u}}{\partial t} = \frac{1}{Re}\left(1+\lambda_3\frac{\partial^\beta}{\partial t^\beta}\right)\left\{-(\xi^2+\zeta_n^2)\bar{u}+\sqrt{\frac{2}{\pi}}\frac{\xi}{\zeta_n}[1-(-1)^n]\right\}$$
$$-\left(1+\lambda_1\frac{\partial^\alpha}{\partial t^\alpha}+\lambda_2\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(M+v\frac{\phi}{K}\right)\bar{u} \qquad (34)$$

$$\bar{u}(\xi,\zeta_n,0) = \frac{\partial\bar{u}(\xi,\zeta_n,0)}{\partial t} = \frac{\partial^2\bar{u}(\xi,\zeta_n,0)}{\partial t^2} = 0 \qquad (35)$$

Let $\bar{\bar{u}}(\xi,\zeta_n,s)$ be the Laplace transform [8] of $\bar{u}(\xi,\zeta_n,t)$ defined by

$$\bar{\bar{u}}(\xi,\zeta_n,s) = \int_0^\infty e^{-st}\,\bar{u}(\xi,\zeta_n,t)\;dt \qquad (36)$$

Now applying Laplace transform to Eq.(34) and taking into initial condition (35) ,we find that

$$\bar{\bar{u}}(\xi,\zeta_n,s) = \sqrt{\frac{2}{\pi}}\frac{\xi}{\zeta_n}\frac{[1-(-1)^n]Re^{-1}s^{-1}(1+\lambda_3\,\mathrm{sign}(1-\beta)s^\beta)}{\left(s+M+v\frac{\phi}{K}\right)[1+\lambda_1 s^\alpha+\lambda_2 s^{2\alpha}]+Re^{-1}(\xi^2+\zeta_n^2)(1+\lambda_3 s^\beta)} \qquad (37)$$

Where

$$\mathrm{sign}(1-\beta) = \begin{cases} -1, & \beta > 1 \\ 0, & \beta = 1 \\ 1, & 0 < \beta < 1 \end{cases} \qquad (38)$$

In order to obtain $\bar{u}(\xi,\zeta_n,t) = L^{-1}(\bar{\bar{u}}(\xi,\zeta_n,s))$ with L^{-1} as the inverse Laplace transform[11] ,for a more suitable presentation of the final results, we rewrite Eq. (37) in the equivalent form

$$\bar{\bar{u}}(\xi,\zeta_n,s) = \sqrt{\frac{2}{\pi}}\frac{\xi\,[1-(-1)^n]}{\zeta_n(\xi^2+\zeta_n^2)}\left[\frac{1}{s}-\frac{1}{s+Re^{-1}(\xi^2+\zeta_n^2)}\right]-\sqrt{\frac{2}{\pi}}\frac{\xi\,[1-(-1)^n]}{\zeta_n(\xi^2+\zeta_n^2)}\frac{(\xi^2+\zeta_n^2)}{Re}$$

$$\frac{1}{s+Re^{-1}(\xi^2+\zeta_n^2)}\left\{[\lambda_1 s^{\alpha+1}+\lambda_2 s^{2\alpha+1}+\left(M+v\frac{\phi}{K}\right)(1+\lambda_1 s^\alpha+\lambda_2 s^{2\alpha})\right.$$
$$+Re^{-1}\lambda_3(\xi^2+\zeta_n^2)(1-\lambda_3\,\mathrm{sign}(1-\beta))s^\beta-\lambda_3\,\mathrm{sign}(1-\beta)s^{\beta+1}]/$$

$$\left\{s\left[\left(s+M+v\frac{\phi}{K}\right)(1+\lambda_1 s^\alpha+\lambda_2 s^{2\alpha})+Re^{-1}(\xi^2+\zeta_n^2)(1+\lambda_3 s^\beta)\right]\right\} \qquad (39)$$

Now , rewriting Eq. (39) in series form as

$$\bar{\bar{u}}(\xi,\zeta_m,s)=\sqrt{\frac{2}{\pi}}\frac{\xi\,[1-(-1)^n]}{\zeta_m\,(\xi^2+\zeta_m^2)}\left[\frac{1}{s}-\frac{1}{s+Re^{-1}(\xi^2+\zeta_m^2)}\right]-\sqrt{\frac{2}{\pi}}\frac{\xi\,[1-(-1)^n]}{\zeta_m\,(\xi^2+\zeta_m^2)}\frac{(\xi^2+\zeta_m^2)}{Re}$$

$$\frac{1}{s+Re^{-1}(\xi^2+\zeta_m^2)}\Big\{\Big[\lambda_1 s^{\alpha+1}+\lambda_2 s^{2\alpha+1}+\left(M+\upsilon\frac{\phi}{K}\right)(1+\lambda_1 s^{\alpha}+\lambda_2 s^{2\alpha})$$

$$+Re^{-1}\lambda_3\,(\xi^2+\zeta_m^2)(1-\lambda_3\,\mathrm{sign}(1-\beta))s^{\beta}-\lambda_3\,\mathrm{sign}(1-\beta)s^{\beta+1}\Big]$$

$$\sum_{k=0}^{\infty}(-1)^k\sum_{i,j,l,d,n,m,w\geq 0}^{i+j+l+d+n+m+w=k}(-1)^i\frac{\left(Re^{-1}(\xi^2+\zeta_m^2)\right)^{n+m}}{i!\,j!\,l!\,d!\,n!\,m!\,w!}$$

$$\left(M+\upsilon\frac{\phi}{K}\right)^{i+w+l+d}\frac{(\lambda_1)^l(\lambda_3)^m}{(\lambda_2)^{i+k+1}}\frac{k!\,s^{\varphi}}{\left[\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+s^{\alpha}\right]^{k+1}}\qquad(40)$$

where $\varphi=\alpha i-j-k+(\alpha-1)l-\alpha k+(\beta-1)m+(2\alpha-1)d-n-2.$

Hence , the Eq.(40) can be written under the form of series as

$$\bar{\bar{u}}(\xi,\zeta_m,s)=\sqrt{\frac{2}{\pi}}\frac{\xi\,[1-(-1)^n]}{\zeta_m\,(\xi^2+\zeta_m^2)}\left[\frac{1}{s}-\frac{1}{s+Re^{-1}(\xi^2+\zeta_m^2)}\right]-\sqrt{\frac{2}{\pi}}\frac{\xi\,[1-(-1)^n]}{\zeta_m\,(\xi^2+\zeta_m^2)}$$

$$\frac{1}{s+Re^{-1}(\xi^2+\zeta_m^2)}\sum_{k=0}^{\infty}(-1)^k\sum_{i,j,l,d,n,m,w\geq 0}^{i+j+l+d+n+m+w=k}(-1)^i\frac{\left(Re^{-1}(\xi^2+\zeta_m^2)\right)^{n+m+1}}{i!\,j!\,l!\,d!\,n!\,m!\,w!}$$

$$\left(M+\upsilon\frac{\phi}{K}\right)^{i+w+l+d}\frac{(\lambda_1)^l(\lambda_3)^m}{(\lambda_2)^{i+k+1}}\left\{\lambda_1\frac{k!\,s^{\varphi+\alpha+1}}{\left[\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+s^{\alpha}\right]^{k+1}}+\lambda_2\right.$$

$$\frac{k!\,s^{\varphi+2\alpha+1}}{\left[\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+s^{\alpha}\right]^{k+1}}+\left(M+\upsilon\frac{\phi}{K}\right)\frac{k!\,s^{\varphi}}{\left[\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+s^{\alpha}\right]^{k+1}}+\lambda_1$$

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$$\left(M+\upsilon\frac{\phi}{K}\right)\frac{k!\ s^{\varphi+\alpha}}{\left[\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+s^{\alpha}\right]^{k+1}}+\lambda_2\left(M+\upsilon\frac{\phi}{K}\right)\frac{k!\ s^{\varphi+2\alpha}}{\left[\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+s^{\alpha}\right]^{k+1}}$$

$$+Re^{-1}\lambda_3(\xi^2+\zeta_m^2)\varepsilon\frac{k!\ s^{\varphi+\beta}}{\left[\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+s^{\alpha}\right]^{k+1}}-\lambda_3\,\mathrm{sign}(1-\beta)$$

$$\left.\frac{k!\ s^{\varphi+2\alpha+\beta+1}}{\left[\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+s^{\alpha}\right]^{k+1}}\right\} \tag{41}$$

Where $\varepsilon=(1-\mathrm{sign}(1-\beta))$

Now , applying the inversion formula term by term for the Laplace transform .And taking convolution integral [10] ,Eq.(41) yields

$$\bar{u}(\xi,\zeta_m,t)=\sqrt{\frac{2}{\pi}}\frac{\xi\ [1-(-1)^n]}{\zeta_m\ (\xi^2+\zeta_m^2)}\left[1-e^{\left(\frac{(\xi^2+\zeta_m^2)}{Re}\right)t}\right]-\sqrt{\frac{2}{\pi}}\frac{\xi\ [1-(-1)^n]}{\zeta_m\ (\xi^2+\zeta_m^2)}\int_0^t e^{\left(\frac{(\xi^2+\zeta_m^2)}{Re}\right)(t-\tau)}$$

$$\sum_{k=0}^{\infty}(-1)^k\sum_{i,j,l,d,n,m,w\geq 0}^{i+j+l+d+n+m+w=k}(-1)^i\frac{\left(Re^{-1}(\xi^2+\zeta_m^2)\right)^{n+m+1}}{i!\,j!\,l!\,d!\,n!\,m!\,w!}\left(M+\upsilon\frac{\phi}{K}\right)^{i+w+l+d}$$

$$\frac{(\lambda_1)^l(\lambda_3)^m}{(\lambda_2)^{i+k+1}}\left\{\lambda_1\tau^{\alpha k-\varphi-2}E^{(k)}_{\alpha,-(\varphi+1)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)+\lambda_2\tau^{\alpha k-\varphi-\alpha-2}\right.$$

$$E^{(k)}_{\alpha,-(\varphi+\alpha+1)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)+\left(M+\upsilon\frac{\phi}{K}\right)\tau^{\alpha k+(\alpha-\varphi)-1}$$

$$E^{(k)}_{\alpha,(\alpha-\varphi)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)+\lambda_1\left(M+\upsilon\frac{\phi}{K}\right)\tau^{\alpha k-\varphi-1}$$

$$E^{(k)}_{\alpha,-\varphi}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)+\lambda_2\left(M+\upsilon\frac{\phi}{K}\right)\tau^{\alpha k-(\alpha+\varphi)-1}$$

$$E^{(k)}_{\alpha,-(\varphi+\alpha)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)+Re^{-1}\lambda_3(\xi^2+\zeta_m^2)\varepsilon$$

$$\tau^{\alpha k+(\alpha-(\varphi+\beta)-1}E^{(k)}_{\alpha,\alpha-(\varphi+\beta)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)-\lambda_3\operatorname{sign}(1-\beta)$$

$$\tau^{\alpha k+(\alpha-\varphi-\beta-2)}E^{(k)}_{\alpha,\alpha-(\varphi+\beta+1)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)\Bigg\}\,d\tau \qquad (42)$$

Where $E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n+\mu)}\quad \lambda,\mu>0$ (43)

represent the generalized Mittag –Leffler function with

$$E^{(k)}_{\lambda,\mu}=\frac{d^k}{dz^k}E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{(n+k)z^n}{n!\,\Gamma(\lambda n+\lambda k+\mu)} \qquad (44)$$

and $\Gamma(.)$ is the Gamma function.

In obtaining Eq.(44) , we used the following property of the generalized Mittag –Leffler function[8]

$$L^{-1}\left\{\frac{k!\,s^{\lambda-\mu}}{(s^{\lambda}\mp c)^{k+1}}\right\}=t^{\lambda k+\mu-1}E^{(k)}_{\lambda,\mu}(\pm ct^{\lambda}),\qquad (\operatorname{Re}(s)>|c|^{1/\lambda}). \qquad (45)$$

Finally, inverting Eq.(43) by means of the mixed Fourier sine formula ,we find the velocity field is given by

$$u(y,z,t)=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin z\,\zeta_N}{\zeta_N}\int_0^{\infty}\frac{\xi\sin y\xi}{(\xi^2+\zeta_N^2)}\left[1-e^{\left(\frac{(\xi^2+\zeta_N^2)}{Re}\right)t}\right]d\xi-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin z\,\zeta_N}{\zeta_N}\int_0^{\infty}\frac{\xi\sin y\xi}{(\xi^2+\zeta_N^2)}$$

$$\int_0^t e^{\left(\frac{(\xi^2+\zeta_N^2)}{Re}\right)(t-\tau)}\sum_{k=0}^{\infty}\sum_{i,j,l,d,n,m,w\geq 0}^{i+j+l+d+n+m+w=k}(-1)^{k+i}\frac{\left(Re^{-1}(\xi^2+\zeta_N^2)\right)^{n+m+1}}{i!\,j!\,l!\,d!\,n!\,m!\,w!}$$

$$\left(M+\upsilon\frac{\phi}{K}\right)^{i+w+l+d}\frac{(\lambda_1)^l(\lambda_3)^m}{(\lambda_2)^{i+k+1}}\left\{\lambda_1\tau^{\alpha k-\varphi-2}E^{(k)}_{\alpha,-(\varphi+1)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)\right.$$

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$$+\lambda_2 \tau^{\alpha k-\varphi-\alpha-2} E^{(k)}_{\alpha,-(\varphi+\alpha+1)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)+\left(M+\upsilon\frac{\phi}{K}\right)$$

$$\tau^{\alpha k+(\alpha-\varphi)-1} E^{(k)}_{\alpha,(\alpha-\varphi)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)+\lambda_1\left(M+\upsilon\frac{\phi}{K}\right)\tau^{\alpha k-\varphi-1}$$

$$E^{(k)}_{\alpha,-\varphi}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)+\lambda_2\left(M+\upsilon\frac{\phi}{K}\right)\tau^{\alpha k-(\alpha+\varphi)-1}$$

$$E^{(k)}_{\alpha,-(\varphi+\alpha)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)+Re^{-1}\lambda_3(\xi^2+\zeta_N^2)\,\varepsilon\,\tau^{\alpha k+(\alpha-(\varphi+\beta)-1}$$

$$E^{(k)}_{\alpha,\alpha-(\varphi+\beta)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)-\lambda_3\,\mathrm{sign}(1-\beta)\,\tau^{\alpha k+(\alpha-\varphi-\beta-2)}$$

$$E^{(k)}_{\alpha,\alpha-(\varphi+\beta+1)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^{\alpha}\right)\Bigg\}\,d\tau d\xi \qquad (46)$$

Where $\zeta_N = (2n-1)\pi$

In which of the following integral is used [8]

$$\int_0^{\infty}\frac{\xi\,\sin y\xi}{(\xi^2+a^2)}d\xi=\frac{\pi}{2}e^{-ay} \qquad (47)$$

$$\int_0^{\infty}\frac{\xi\,\sin y\xi}{(\xi^2+b^2)}e^{-a\xi^2}\,d\xi=\frac{\pi}{4}e^{ab^2}\left[e^{-by}erfc\left(b\sqrt{a}-\frac{y}{2\sqrt{a}}\right)-e^{by}erfc\left(b\sqrt{a}-\frac{y}{2\sqrt{a}}\right)\right] \quad (48)$$

Where $erfc(.)$ is the complementary error function and using Eqs.(47) and (48) into Eq(46) ,we get

$$u(y,z,t)=4\sum_{n=1}^{\infty}\frac{\sin z\zeta_N}{\zeta_N}e^{-\zeta_N y}-2\sum_{n=1}^{\infty}\frac{\sin z\zeta_N}{\zeta_N}e^{\left(\frac{(\xi^2+\zeta_N^2)}{Re}\right)t}\{e^{-\zeta_N y}$$

$$erfc\left(\zeta_N\sqrt{\frac{t}{Re}}-\frac{y}{2\sqrt{t/Re}}\right)-e^{\zeta_N y}\;erfc\left(\zeta_N\sqrt{\frac{t}{Re}}-\frac{y}{2\sqrt{t/Re}}\right)\Bigg\}-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin z\zeta_N}{\zeta_N}$$

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$$\int_0^\infty \frac{\xi \sin y\xi}{(\xi^2+\zeta_N^2)}\int_0^t e^{\left(\frac{(\xi^2+\zeta_N^2)}{Re}\right)(t-\tau)}\sum_{k=0}^{\infty}\sum_{i,j,l,d,n,m,w\geq 0}^{i+j+l+d+n+m+w=k}(-1)^{k+i}\frac{\left(Re^{-1}(\xi^2+\zeta_N^2)\right)^{n+m+1}}{i!\,j!\,l!\,d!\,n!\,m!\,w!}$$

$$\left(M+\upsilon\frac{\phi}{K}\right)^{i+w+l+d}\frac{(\lambda_1)^l(\lambda_3)^m}{(\lambda_2)^{i+k+1}}\left\{\lambda_1\tau^{\alpha k-\varphi-2}E^{(k)}_{\alpha,-(\varphi+1)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^\alpha\right)\right.$$

$$+\lambda_2\tau^{\alpha k-\varphi-\alpha-2}E^{(k)}_{\alpha,-(\varphi+\alpha+1)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^\alpha\right)+\left(M+\upsilon\frac{\phi}{K}\right)$$

$$\tau^{\alpha k+(\alpha-\varphi)-1}E^{(k)}_{\alpha,(\alpha-\varphi)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^\alpha\right)+\lambda_1\left(M+\upsilon\frac{\phi}{K}\right)\tau^{\alpha k-\varphi-1}$$

$$E^{(k)}_{\alpha,-\varphi}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^\alpha\right)+\lambda_2\left(M+\upsilon\frac{\phi}{K}\right)\tau^{\alpha k-(\alpha+\varphi)-1}$$

$$E^{(k)}_{\alpha,-(\varphi+\alpha)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^\alpha\right)+Re^{-1}\lambda_3(\xi^2+\zeta_N^2)\,\varepsilon\,\tau^{\alpha k+(\alpha-(\varphi+\beta)-1}$$

$$E^{(k)}_{\alpha,\alpha-(\varphi+\beta)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^\alpha\right)-\lambda_3\,\mathrm{sign}(1-\beta)\,\tau^{\alpha k+(\alpha-\varphi-\beta-2)}$$

$$\left.E^{(k)}_{\alpha,\alpha-(\varphi+\beta+1)}\left(-\left(\frac{\left(M+\upsilon\frac{\phi}{K}\right)}{\lambda_2}+\frac{\lambda_1}{\lambda_2}\right)\tau^\alpha\right)\right\}d\tau d\xi \tag{49}$$

the velocity field $u(y,z,t)$ for a fractional Burgers' fluid given by Eq. (49) has two parts: the first part corresponding to a Newtonian fluid performing the same motion and the second part on the right-hand side of Eq.(49) resulting from the viscoelastic property of a fractional Burgers' fluid.

3.4 Special Cases

1- Making the limit of Eq.(49) when $M=0$ and $\upsilon\frac{\phi}{K}=0$ $(i=w=l=d=0)$, we can get similar solution velocity distribution and shear stress for unsteady flows of a viscelastic fluid with the fractional Burgers' model, as obtained in [8]. Thus the velocity field reduces to

$$u(y,z,t)=4\sum_{n=1}^{\infty}\frac{\sin z\zeta_N}{\zeta_N}e^{-\zeta_N y}-2\sum_{n=1}^{\infty}\frac{\sin z\zeta_N}{\zeta_N}e^{\left(\frac{(\xi^2+\zeta_N^2)}{Re}\right)t}\{e^{-\zeta_N y}$$

$$erfc\left(\zeta_N\sqrt{\frac{t}{Re}}-\frac{y}{2\sqrt{t/Re}}\right)-e^{\zeta_N y}\; erfc\left(\zeta_N\sqrt{\frac{t}{Re}}-\frac{y}{2\sqrt{t/Re}}\right)\Bigg\}-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin z\,\zeta_N}{\zeta_N}$$

$$\int_0^{\infty}\frac{\xi\sin y\xi}{(\xi^2+\zeta_N^2)}\int_0^t e^{\left(\frac{(\xi^2+\zeta_N^2)}{Re}\right)(t-\tau)}\sum_{k=0}^{\infty}\sum_{l,n,m\geq 0}^{l+n+m=k}(-1)^k\frac{\left(Re^{-1}(\xi^2+\zeta_N^2)\right)^{n+m+1}}{n!\,m!}\frac{(\lambda_3)^m}{(\lambda_2)^{k+1}}$$

$$\Bigg\{\lambda_1\tau^{\alpha k-\varphi-2}E^{(k)}_{\alpha,-(\varphi+1)}\left(-\frac{\lambda_1}{\lambda_2}\tau^{\alpha}\right)+\lambda_2\tau^{\alpha k-\varphi-2}E^{(k)}_{\alpha,-(\varphi+\alpha+1)}\left(-\frac{\lambda_1}{\lambda_2}\tau^{\alpha}\right)$$

$$+Re^{-1}\lambda_3(\xi^2+\zeta_N^2)\,\varepsilon\,\tau^{\alpha k+(\alpha-(\varphi+\beta)-1}E^{(k)}_{\alpha,\alpha-(\varphi+\beta)}\left(-\frac{\lambda_1}{\lambda_2}\tau^{\alpha}\right)$$

$$-\lambda_3\,\mathrm{sign}(1-\beta)\,\tau^{\alpha k+(\alpha-\varphi-\beta-2)}\;E^{(k)}_{\alpha,\alpha-(\varphi+\beta+1)}\left(-\frac{\lambda_1}{\lambda_2}\tau^{\alpha}\right)\Bigg\}\,d\tau d\xi \qquad (50)$$

where $\varphi=m+j\beta-k\alpha-\alpha-k-2$

2- Making the limit of Eq.(49) when $M\to 0$, $\upsilon\frac{\phi}{K}\to 0$, $\lambda_1\to 0$ and $\lambda_2\to 0(i=j=w=d=0)$, we can get the velocity distribution for a generalized second -grade fluid obtained by Khan and Wang[12]. Thus the velocity field reduces to

$$u(y,z,t)=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin z\,\zeta_N}{\zeta_N}\int_0^{\infty}\frac{\xi\sin y\xi}{(\xi^2+\zeta_N^2)}\sum_{k=0}^{\infty}(-1)^k\frac{\left(Re^{-1}(\xi^2+\zeta_N^2)\right)^{k+1}}{k!}\Big\{t^{k+1}$$

$$E^{(k)}_{1-\beta,\beta k+2}\left(-\frac{(\xi^2+\zeta_N^2)}{Re}t^{1-\beta}\right)+Re^{-1}\mathrm{sign}(1-\beta)t^{k-\beta+1}$$

$$E^{(k)}_{1-\beta,\beta k-\beta+2}\left(-\frac{(\xi^2+\zeta_N^2)}{Re}t^{1-\beta}\right)\Big\}\,d\xi \qquad (51)$$

3- Making the limit of Eq.(49) when $M\to 0$, $\upsilon\frac{\phi}{K}\to 0$,and $\lambda_2\to 0(i=j=w=d=0)$, we can get the velocity distribution for a generalized Oldroyd-B fluid obtained by Fetecau et al [2].Thus the velocity field reduces to

$$u(y,z,t)=4\sum_{n=1}^{\infty}\frac{\sin z\,\zeta_N}{\zeta_N}e^{-\zeta_N y}-2\sum_{n=1}^{\infty}\frac{\sin z\,\zeta_N}{\zeta_N}e^{\left(\frac{(\xi^2+\zeta_N^2)}{Re}\right)t}\Big\{e^{-\zeta_N y}$$

$$erfc\left(\zeta_N\sqrt{\frac{t}{Re}}-\frac{y}{2\sqrt{t/Re}}\right)-e^{\zeta_N y}\; erfc\left(\zeta_N\sqrt{\frac{t}{Re}}-\frac{y}{2\sqrt{t/Re}}\right)\Bigg\}-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin z\,\zeta_N}{\zeta_N}$$

$$\int_0^\infty \frac{\xi \sin y\xi}{(\xi^2+\zeta_N^2)} \int_0^t e^{\left(\frac{(\xi^2+\zeta_N^2)}{Re}\right)(t-\tau)} \sum_{k=0}^\infty \sum_{m=0}^k (-1)^k \frac{\left(Re^{-1}(\xi^2+\zeta_N^2)\right)^{k-m+1}}{(\lambda_1)^{k+1} m!\,(k-m)!}$$

$$\left\{\lambda_1 \tau^\varphi E^{(k)}_{\theta+1,\gamma}\left(-\frac{\lambda_3(\xi^2+\zeta_N^2)}{\lambda_1 Re}\tau^\alpha\right) - \lambda_3\,\mathrm{sign}(1-\beta)\tau^{\varphi+\theta} E^{(k)}_{\theta+1,\theta+\gamma}\left(-\frac{\lambda_3(\xi^2+\zeta_N^2)}{\lambda_1 Re}\tau^\alpha\right)\right.$$

$$+Re^{-1}\lambda_3(\xi^2+\zeta_N^2)\,\varepsilon\,\tau^{\varphi+\theta+1} E^{(k)}_{\theta+1,\theta+\gamma+1}\left(-\frac{\lambda_3(\xi^2+\zeta_N^2)}{\lambda_1 Re}\tau^\alpha\right) \qquad (52)$$

Where $\varphi = k\alpha + k - m$ and $\theta = \alpha - \beta$

4- Making the limit of Eq.(49) when $M \to 0$, $\upsilon\frac{\phi}{K} \to 0$, $\lambda_2 \to 0$ and $\lambda_3 \to 0 (i = j = w = d = 0)$, we can get the velocity distribution for a generalized Maxwell fluid obtained by Vieru et al [3]. Thus the velocity field reduces to

$$u(y,z,t) = \frac{8}{\pi} \sum_{n=1}^\infty \frac{\sin z\zeta_N}{\zeta_N} \int_0^\infty \frac{\xi \sin y\xi}{(\xi^2+\zeta_N^2)} \sum_{k=0}^\infty (-1)^k \frac{\left(Re^{-1}(\xi^2+\zeta_N^2)\right)^{k+1}}{(\lambda_1)^{k+1} k!} \tau^{\alpha k+\alpha+k+1}$$

$$E^{(k)}_{\alpha,\alpha+k+2}\left(-\frac{\tau^\alpha}{\lambda_1}\right)\Big\}\, d\xi \qquad (53)$$

3.5 Numerical Results and Discussion:

In this work, we have discussed the effect of the MHD flow of an incompressible generalized fractional Burgers' fluid between two side walls perpendicular to the plate in porous medium. The exact solution for the velocity field u is obtained by using the discrete Laplace and mixed Fourier sine transforms. Moreover, some figures are plotted to show the behavior of various parameters involved in the expressions of velocity u .

All the results in this section are made through plotting graph by using MATHEMATICA package.

Based on Eq.(58), Figs.(2 - 13) , illustrates the effects of the fractional parameters (α and β) , Renold number Re , relaxation time λ_1 , retardation time λ_3 , material parameter of the Burgers' fluid λ_2 , time t , parameter y , magnetic number M , the porosity of porous medium ϕ , permeability K and υ is the kinematics' viscosity on the velocity .

Fig.(2 , 3) , illustrate the effects of non-integer fractional parameters (α and β) on the velocity fields . It is observed that the velocity is decreasing with the increased the α and β Fig. (4) , is established to show the behavior of Renold number Re , The velocity is decreasing with the increase of Renold number Re . Fig. (5) , contains the behavior of u under the variation of relaxation time λ_1 , one can depict here that *u* increase with the increasing effects of the parameter relaxation time λ_1 .

Fig.(6) , provides the graphical illustration for the effect of material parameter of the Burgers' fluid λ_2 on the velocity fields. The velocity is decreasing with the increase of λ_2 . Fig.(7) , are prepared to show the effect of the retardation time λ_3 on the velocity field. The velocity is decreasing with the increase of the retardation time λ_3 . Fig.(8) , is established to show the behavior of parameter y on the velocity field , it is observed that the velocity is

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decreased with increase the parameter y. Fig.(9), displays the variations of velocity with z at different magnetic

number M . It is interestingly observed that the velocity u rising up with the increasing effects of the parameter magnetic number M . Fig .(10) illustrates the effects of the kinematics' viscosity υ on the velocity function ,it is found that the velocity u decreasing with the increasing effects of the kinematics' viscosity υ . Fig .(11) it is observed that increase in permeability K results in increase of velocity field . Fig .(12) , provides the graphical illustration for the effect of ϕ on the velocity field . the velocity is decreased with increase of ϕ . Fig.(13), is prepared to show the effect of retardation time t on the velocity fields. The velocity is increased with the increase of time t .

3.6 Conclusion

We have discussed the effect the MHD flow of an incompressible generalized fractional Burgers' fluid between two side walls perpendicular to the plate in porous medium, yield the effect of each parameter upon the velocity field will be considered. The effect of any parameter, this parameter will be taken in some range, while the other parameters will kept fixed.

The following results are observed:

1- As α and β increasing there is decreasing in the velocity profile. Figures (2) and (3).

decreasing there is increase in the velocity profile. Figures (4), The Reynolds Re As -2

number Re is an important dimensionless parameter defining the laminar or turbulent flow. It is well known that the thickness of boundary layer is inversely

proportional to the value of Re.

1- As λ_2 , y , υ and ϕ decreasing there is increasing in the velocity profile. Figs. (6) ,(8) ,(10) ,and (12).
2- As λ_1, λ_3 , M , K and t increasing there is decreasing in the velocity profile. Figure (5),(7), (9) ,(11) and (13).

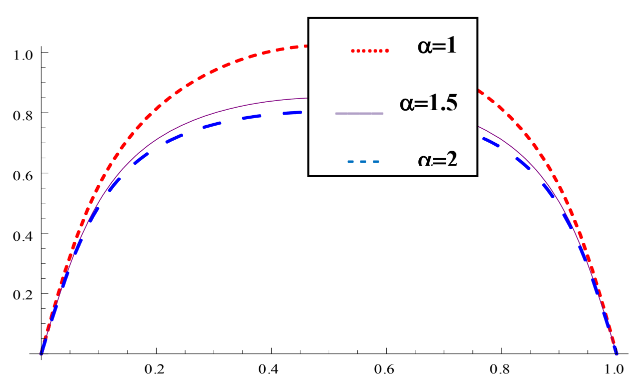

Fig.(2):- the velocity u for different value , keeping other parameters fixed
{ 7 , Re 1 ,K 0.001, λ_2 2 , λ_1 5 , λ_3 1 , 0.8 , t 0.3 , 0.00001 , y 0.1 , M 0.1}

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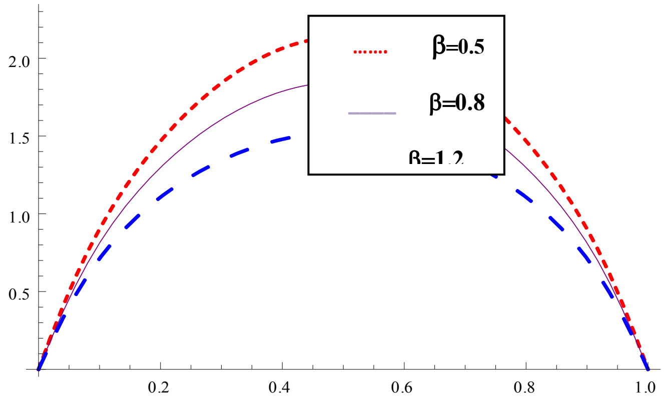

Fig.(3):- the velocity u for different value , keeping other parameters fixed
{ 7 , Re 1 ,K 0.001, λ_2 2 , λ_1 5 , λ_3 1 , 0.8 , t 0.3 , 0.00001 , y 0.1 , M 0.1}

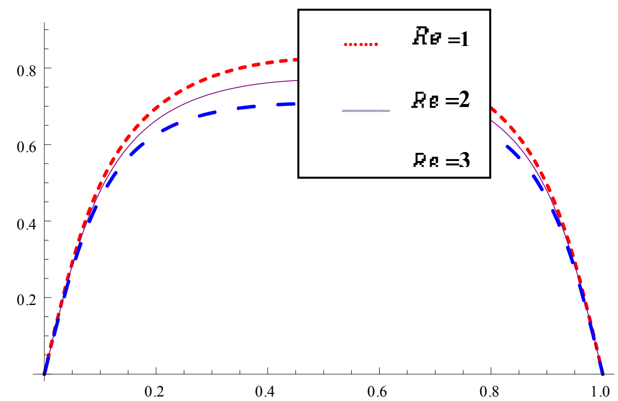

Fig.(4):- the velocity u for different value Re , keeping other parameters fixed
{ 7 , 2 ,K 0.001, λ_2 2 , λ_1 5 , λ_3 1 , 0.8 , t 0.3 , 0.00001 , y 0.1 , M 0.1}

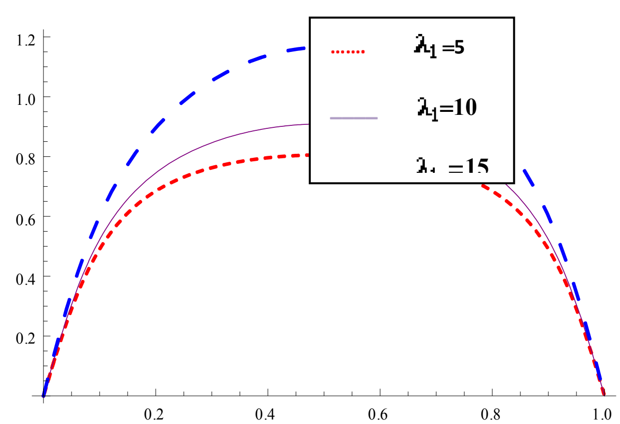

Fig.(5):- the velocity u for different value λ_1 , keeping other parameters fixed
{ 7 , Re 1 ,K 0.001, 2 , λ_2 5 , λ_3 1 , 0.8 , t 0.3 , 0.00001 , y 0.1 , M 0.1}

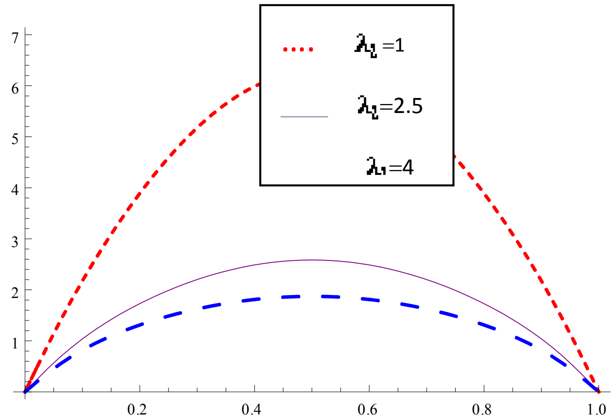

Fig.(6):- the velocity u for different value λ_1 , keeping other parameters fixed
{ 7 , Re 1 ,K 0.001, 2 , λ_1 2 , λ_3 1 , 0.8 , t 0.3 , 0.00001 , y 0.1 , M 0.1}

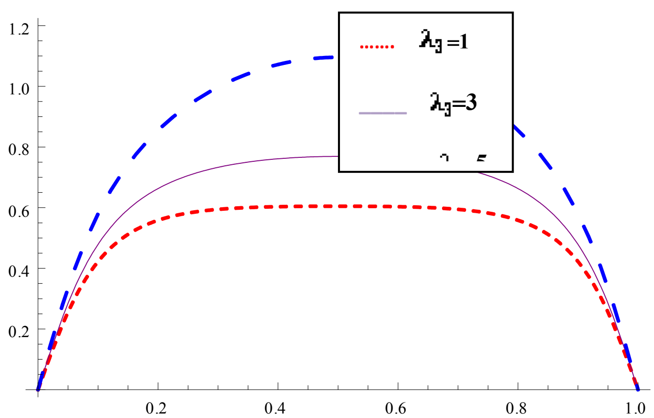

Fig .(7):- the velocity u for different value λ_3 , keeping other parameters fixed
{M 0.1 , Re 1 , t 0.3 , 2 , λ_2 2 , 0.00001 , 0.8 , K 0.001 , 7 , λ_1 5 , y 0.1}

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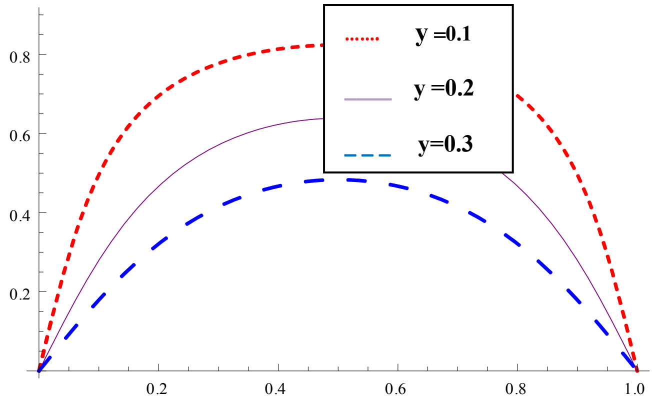

Fig.(8):- the velocity u for different value y *, keeping other parameters fixed*
{ 7 , Re 1 , K 0.001, 2 , λ_2 *2 ,* λ_3 *1 , 0.8 , t 0.3 , 0.00001 ,* λ_1 *5 ,*
M 0.1}

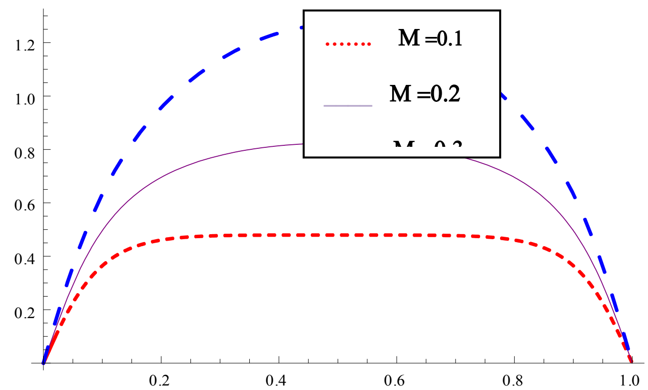

Fig.(9):- the velocity u for different value M *, keeping other parameters fixed*
{ 7 , Re 1 , K 0.001, 2 , λ_2 *2 ,* λ_3 *1 , 0.8 , t 0.3 , 0.00001 ,* λ_1 *5 , y 0.1}*

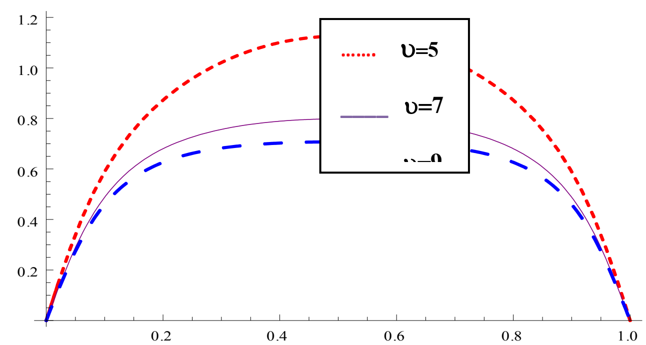

Fig.(10):- the velocity u for different value , keeping other parameters fixed
{M 0.1 , Re 1 , K 0.001, 2 , λ_2 *2 ,* λ_3 *1 , 0.8 , t 0.3 , 0.00001 ,* λ_1 *5 ,*
y 0.1}

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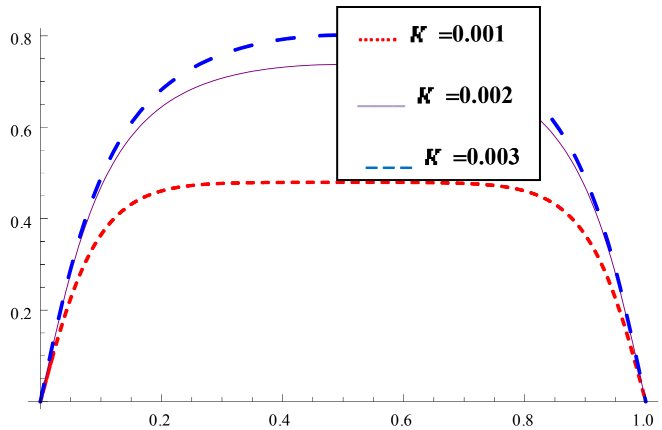

Fig.(11):- the velocity u for different value K , keeping other parameters fixed
{M 0.1 , Re 1 , 0.00001, 2 , λ_1 2 , λ_3 1 , 0.8 , t 0.3 , 7 , λ_1 5 , y 0.1}

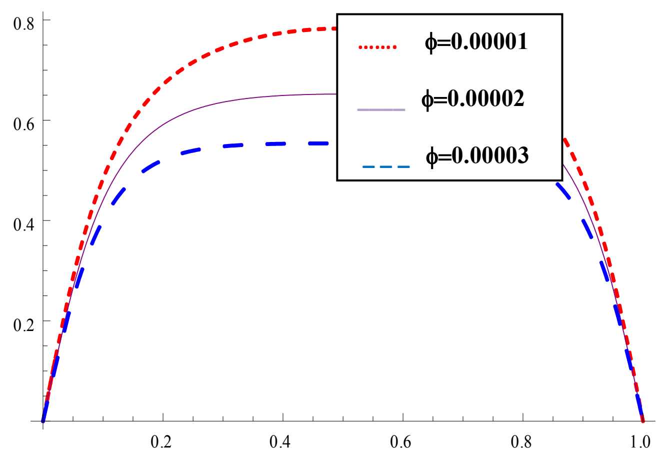

Fig.(12):- the velocity u for different value , keeping other parameters fixed
{M 0.1 , Re 1 , t 0.3, 2 , λ_1 2 , λ_3 1 , 0.8 , K 0.001 , 7 , λ_1 5 , y 0.1}

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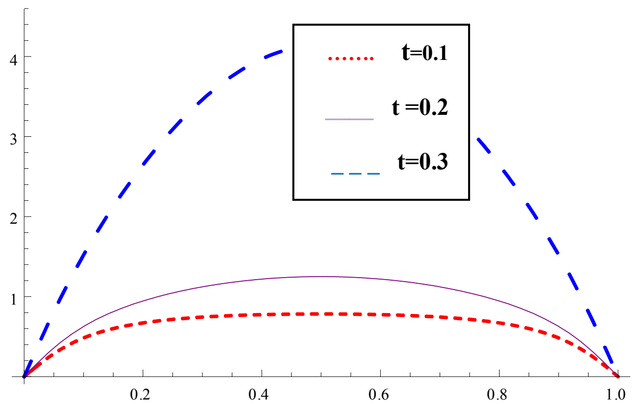

Fig.(13):- the velocity u for different value t , keeping other parameters fixed
{M 7 , Re 1 , K 0.001, 2 , λ_2 2 , λ_3 1 , 0.8 , 0.00001 , λ_1 5 , y 0.1}

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